

# Solutions of $D = 2$ supersymmetric Yang-Mills quantum mechanics with $SU(N)$ gauge group

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## Abstract

We describe the generalization of the recently derived solutions of  $D = 2$  supersymmetric Yang-Mills quantum mechanics with  $SU(3)$  gauge group to the generic case of  $SU(N)$  gauge group. We discuss the spectra and eigensolutions in bosonic as well as fermionic sectors.

## 1 Introduction

Supersymmetric Yang-Mills quantum mechanics [1, 2] attract a lot of attention since the works Ref.[3, 4]. Among many variants of such quantum mechanics, the twodimensional systems are the simplest ones. Although proposed nearly 30 years ago by Claudson and Halpern [2], their solutions for gauge groups other than  $SU(2)$  are poorly known. Recently, complete solutions were derived for the model with the  $SU(3)$  gauge group [5]. In this work we present a generalization of such solutions to the case of models with any  $SU(N)$  gauge group. We discuss their derivation as well as their properties. We use the framework of cut Fock basis described in details in Ref.[6], which allows a systematic analysis of the SYMQM systems by both numerical and analytic methods.

The cut Fock space approach basically consists in introducing the Fock basis in the Hilbert space and then in considering only a finite subset of basis states. This subset is composed of states with less than  $N_{cut}$  quanta and  $N_{cut}$  is usually called the cut-off.

The main results presented in this paper are the closed formulae for the eigenenergies and corresponding eigenstates of SYMQM in any, bosonic or fermionic, sector, valid for any  $N_{cut}$ . Their infinite cut-off limit is also discussed.

Such results are especially interesting since they enable one to study the large- $N$  behavior of the wavefunctions of SYMQM as well as the supersymmetric structure of the limiting model. They may be used to construct higher dimensional wave-functions which can have an interpretation in the context of supermembrane theory [7]. One may also try to build a perturbation theory around their finite cut-off versions [6].

This paper is composed in the following way. We start by briefly presenting the  $D = 2$  supersymmetric Yang-Mills quantum mechanics and the approach of cut Fock basis. Then, we translate the eigenequation for the Hamiltonian to a recursion relation for the coefficients describing the decomposition of the eigenstate in the Fock basis. The recursion relation is valid for any  $N$ . Subsequently, we discuss the implications of this recurrence relation. Specifically, we argue that a closed formula for the spectra can be deduced in all, bosonic as well as fermionic, sectors of the SYMQM models. We explicitly present it for the  $SU(4)$  and  $SU(5)$  models. We also mention the form of the spectra in the infinite cut-off limit. In section 7 expressions for the eigenstates are presented. The emphasis is put on the wave-functions of supersymmetric vacua in a generic  $SU(N)$  model. Next, the completeness, orthogonality, normalization and infinite cut-off limit of our eigensolutions are discussed. Finally, we end with some conclusions.

## 2 Supersymmetric Yang-Mills Quantum Mechanics

Although supersymmetric Yang-Mills quantum mechanics were described already many times [2, 8, 9, 6], let us briefly remind the main statements in order to keep this work self-contained. SYMQM can be obtained by a dimensional reduction of a supersymmetric,  $D = d + 1$  dimensional Yang-Mills quantum field theory to  $D = 0 + 1$ , i.e. to a single point in space. The remaining degrees of freedom are those of internal symmetries of the field theory. Consequently, the initial local gauge symmetry is reduced to a global symmetry of the quantum mechanical system. In this work we will be interested in systems obtained from  $\mathcal{N} = 1$  Yang-Mills field theory in two dimensions with different  $SU(N)$  gauge symmetries [2]. The degrees of freedom are described by a scalar field  $\phi_A$  and a complex fermion  $\lambda_A$ , where  $A$  labels the generators of the gauge group, hence  $\phi$  and  $\lambda$  transform in the adjoint representation of  $SU(N)$ . The Hamiltonian of the reduced system

reads <sup>1</sup>

$$H = \frac{1}{2}\pi_A\pi_A + igf_{ABC}\bar{\lambda}_A\phi_B\lambda_C. \quad (1)$$

$H$  is supersymmetric since we can define the supercharges  $Q$  and  $\bar{Q}$ ,

$$Q = \lambda_A\pi_A, \quad \bar{Q} = \bar{\lambda}_A\pi_A, \quad (2)$$

such that

$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0, \quad \text{and} \quad \{Q, \bar{Q}\} = \pi_A\pi_A - 2g\phi_A G_A, \quad (3)$$

where

$$G_A = f_{ABC}(\phi_B\pi_C - i\bar{\lambda}_B\lambda_C), \quad (4)$$

is the generator of the gauge transformations. Upon canonical quantization, one defines operators satisfying canonical commutation/anticommutation relations,

$$[\phi_A, \pi_B] = i\delta_{A,B}, \quad \{\lambda_A, \bar{\lambda}_B\} = \delta_{A,B}. \quad (5)$$

The quantization procedure requires the imposition of the Gauss' constraint on physical states. The dimensionally reduced Gauss' law translates to a requirement of invariance of physical states under gauge transformations,

$$G_A|\text{ physical state } \rangle = 0. \quad (6)$$

Thus, the fermionic term of the Hamiltonian, being proportional to the Gauss' constraint, vanishes on any physical state. Therefore,

$$H = \{Q, Q^\dagger\} = \frac{1}{2}\pi_A\pi_A \quad \text{in the physical Hilbert space.} \quad (7)$$

We end this section by rewriting the above Hamiltonian in terms of creation and annihilation operators, defined as

$$a_A = \frac{1}{\sqrt{2}}(\phi_A + i\pi_A), \quad a_A^\dagger = \frac{1}{\sqrt{2}}(\phi_A - i\pi_A). \quad (8)$$

Thus,

$$H = \text{tr}(a^\dagger a) + \frac{N^2 - 1}{4} - \frac{1}{2}\text{tr}(a^\dagger a^\dagger) - \frac{1}{2}\text{tr}(aa), \quad (9)$$

where we used the matrix notation, in which every operator transforming in the adjoint representation is summed with the generators of the  $SU(N)$  group in the fundamental representation, giving an operator valued  $N \times N$  matrix. In what follows we will use a simplified notation for the trace of any such matrix, namely,  $\text{tr}(O) \equiv (O)$ .

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<sup>1</sup>The summation of doubled indices is understood, i.e.  $\phi_A\phi_A \equiv \sum_{A=1}^{N^2-1} \phi_A\phi_A$ .

### 3 Gauge invariant Fock basis for $SU(N)$ SYMQM

A systematic construction of the Fock basis for the  $D = 2$  SYMQM models was proposed in Ref.[8] and developed in Ref.[9]. We use the notation introduced in Ref.[6] where the recursive construction of the basis using the notion of elementary bosonic and fermionic bricks was described in details. We recall here very briefly the most important conclusions concerning the Fock basis of the SYMQM Hilbert space.

An operator is called an elementary bosonic brick if it is a single trace operator composed exclusively of creation operators. For a given  $N$  we have  $N - 1$  linearly independent elementary bosonic bricks, which we label by  $C^\dagger$ . They are

$$C_N^\dagger(2) \equiv (a^{\dagger 2}), C_N^\dagger(3) \equiv (a^{\dagger 3}), \dots, C_N^\dagger(N-1) \equiv (a^{\dagger N-1}), C_N^\dagger(N) \equiv (a^{\dagger N}).$$

A generic basis state can be written as

$$|p_2, p_3, \dots, p_N\rangle = C_N^\dagger(2)^{p_2} C_N^\dagger(3)^{p_3} \dots C_N^\dagger(N-1)^{p_{N-1}} C_N^\dagger(N)^{p_N} |0\rangle. \quad (10)$$

Additionally, in the fermionic sector with  $n_F$  fermionic quanta there are  $d^{n_F}(N)$  fermionic bricks. We label them by  $C_N^\dagger(n_B^\alpha, n_F, \alpha)$ , where  $n_B^\alpha$  denotes the number of bosonic creation operators and  $n_F$  the number of fermionic creation operators incorporated in  $C_N^\dagger(n_B^\alpha, n_F, \alpha)$ .  $\alpha$  is an additional index, since  $n_B^\alpha$  and  $n_F$  do not specify unambiguously the operator.  $\alpha$  runs from 1 to  $d^{n_F}(N)$  in each fermionic sector. Fermionic basis states can be obtained by the application of the fermionic bricks to the bosonic basis states eq.(10). Hence, we define

$$|\alpha, n_F; p_2, p_3, \dots, p_N\rangle = C_N^\dagger(n_B^\alpha, n_F, \alpha) |p_2, p_3, \dots, p_N\rangle. \quad (11)$$

Although the sets of fermionic bricks are not explicitly known for  $N > 4$ , it turns out that they are not necessary for the derivation of spectra of the  $SU(N)$  SYMQM models.

A generic state from the bosonic sector with up to  $N_{cut}$  bosonic quanta can be decomposed as

$$|E\rangle = \sum_{2p_2+3p_3+\dots+Np_N \leq N_{cut}} a_{p_2, p_3, \dots, p_N} |p_2, p_3, \dots, p_N\rangle, \quad (12)$$

whereas a generic state from the sector with  $n_F$  fermionic quanta can be decomposed in the so constructed basis with unknown amplitudes  $a_{p_2, p_3, \dots, p_N}^\alpha(E)$ ,

where the index  $\alpha$  describes which one of the fermionic bricks was used. For the cut-off  $N_{cut}$  we get

$$|E\rangle = \sum_{\alpha=1}^{d^{n_F}(N)} \sum_{\sum_{k=2}^N kp_k \leq N_{cut} - n_B^\alpha} a_{p_2, p_3, \dots, p_N}^\alpha |\alpha, n_F; p_2, p_3, \dots, p_N\rangle. \quad (13)$$

The physical results correspond to the limit of  $N_{cut} \rightarrow \infty$ . Such limit is nontrivial in the case of systems with continuous spectra and was discussed in details in the case of a system with one degree of freedom [10] as well as systems with a  $SO(d)$  gauge symmetry [11]. In this paper we will describe the  $N_{cut} \rightarrow \infty$  limit, however, the study of the scaling law in the spirit of Refs.[10, 11] will be discussed elsewhere.

## 4 Recurrence relations

In this section we derive the recurrence relation for the coefficients  $a_{p_2, p_3, \dots, p_N}$  and  $a_{p_2, p_3, \dots, p_N}^\alpha$ . To this goal we follow the derivation of a similar recurrence relations for the  $SU(3)$  model [12].

### 4.1 Bosonic sectors

We start with the purely bosonic sector. In order to obtain the recurrence relation for  $a_{p_2, p_3, \dots, p_N}$  we must evaluate the action of the Hamiltonian on a generic basis state. We get (see Appendix A for the details of calculations):

$$\begin{aligned} (a^\dagger a) |p_2, \dots, p_N\rangle &= \frac{1}{2} \left( \sum_{k=2}^N kp_k \right) |p_2, \dots, p_N\rangle, \\ (a^\dagger a^\dagger) |p_2, \dots, p_N\rangle &= |p_2 + 1, p_3, \dots, p_N\rangle, \\ (aa) |p_2, \dots, p_N\rangle &= \left( p_2(p_2 + \frac{1}{2}(N^2 - 1) - 1 \sum_{k=3}^N kp_k) \right) |p_2 - 1, p_3, \dots, p_N\rangle + \\ &+ \sum_{j=3}^N \left( \frac{j^2 p_j (p_j - 1)}{4} ((a^{\dagger 2j-2}) - \frac{1}{N} (a^{\dagger j-1})^2) + \frac{j p_j}{4} \sum_{t=2}^{j-4} (a^{\dagger t}) (a^{\dagger j}) (a^{\dagger j-2-t}) + \right. \\ &+ \frac{N j p_j}{4} (1 - \frac{j-1}{N^2}) (a^{\dagger j}) (a^{\dagger j-2}) + \sum_{s=j+1}^N \frac{j p_j s p_s}{2} \left( \frac{(a^{\dagger j+s-2}) (a^{\dagger j})}{(a^{\dagger s})} + \right. \\ &\left. \left. - \frac{1}{N} \frac{(a^{\dagger j-1}) (a^{\dagger j}) (a^{\dagger s-1})}{(a^{\dagger s})} \right) \right) |p_2, \dots, p_j - 2, \dots, p_N\rangle. \end{aligned} \quad (14)$$

Eqs.(14) lead to the general recursion relation

$$\begin{aligned}
& a_{p_2-1, \dots, p_N} - \left( \sum_{k=2}^N k p_k + \frac{1}{2}(N^2 - 1) - 2E \right) a_{p_2, \dots, p_N} + \\
& + \left( (p_2 + 1)(p_2 + \frac{1}{2}(N^2 - 1) + \sum_{k=3}^N k p_k) \right) a_{p_2+1, \dots, p_N} + \\
& + \sum_{j=3}^N \left( \frac{p_j(p_j - 1)}{4} j^2 \left( a_{p_2, \dots, p_j+2, \dots, p_{2j-2}-1, \dots, p_N} - \frac{1}{N} a_{p_2, \dots, p_{j-1}-2, p_j+2, \dots, p_N} \right) + \right. \\
& + p_j j \frac{N}{4} \left( 1 - \frac{j-1}{N^2} \right) a_{p_2, \dots, p_{j-2}-1, p_{j-1}, p_j+1, \dots, p_N} + \\
& + \frac{p_j j}{4} \sum_{t=2}^{j-4} a_{p_2, \dots, p_t-1, \dots, p_{j-2-t}-1, \dots, p_j+1, \dots, p_N} + \\
& + p_j j \sum_{s=j+1}^N \frac{p_s s}{2} \left( a_{p_2, \dots, p_j+1, \dots, p_s+1, \dots, p_{j+s-2}-1, \dots, p_N} + \right. \\
& \left. \left. - \frac{1}{N} a_{p_2, \dots, p_{j-1}-1, p_j+1, \dots, p_{s-1}-1, p_s+1, \dots, p_N} \right) \right) = 0. \quad (15)
\end{aligned}$$

An important remark concerns the Cayley-Hamilton theorem. In deriving eq.(15) we did not simplified the operators with powers of bosonic creation operators bigger than  $N$ . Such operators can appear in two places in eq.(14), namely for the operators  $(a^{\dagger 2j-2})$  and  $(a^{\dagger j+s-2})$ , where  $3 \leq j \leq N$  and  $j+1 \leq N$ . They must be reduced once the final form of the recurrence relation for a given  $N$  is obtained. Note, however, that when examining the large- $N$  limit, the Cayley-Hamilton theorem does not apply, therefore, our recurrence relation is a good starting point for such investigation.

With  $N = 3$  and after simplifying the operator  $(a^{\dagger 4})$  with the Cayley-Hamilton theorem, we recover the recurrence relation discussed in Refs.[5, 12], namely,

$$\begin{aligned}
& a_{p_2-1, p_3} - (2p_2 + 3p_3 + 4 - 2E) a_{p_2, p_3} + (p_2 + 1)(p_2 + 3p_3 + 4) a_{p_2+1, p_3} + \\
& + \frac{3}{8} (p_3 + 1)(p_3 + 2) a_{p_2-2, p_3+2} = 0. \quad (16)
\end{aligned}$$

The first three terms does not involve any change of the  $p_3$  index. The mixing of amplitudes with different values of the  $p_3$  index is described only by the fourth term. It is induced by the non-orthogonal states  $\langle 0, 2|3, 0 \rangle \neq 0$ .

In order to demonstrate the increasing complexity of the recursion relation with increasing  $N$  induced by the increasing number of mixing terms between states with the same number of bosonic quanta but made with different elementary bricks, let us present the recursion relation for the  $SU(4)$  model, which reads

$$\begin{aligned}
& a_{p_2-1,p_3,p_4} - \left(2p_2 + 3p_3 + 4p_4 + \frac{15}{2} - 2E\right) a_{p_2,p_3,p_4} + \\
& + (p_2 + 1)(p_2 + 3p_3 + 4p_4 + \frac{15}{2}) a_{p_2+1,p_3,p_4} \\
& + \left(5p_3(p_4 + 1) - \frac{3}{2} + 3p_4(p_4 + 1) + \frac{13}{4}(p_4 + 1)\right) a_{p_2-1,p_3,p_4+1} + \\
& + \frac{1}{3}(p_4 + 1)(p_4 + 2) a_{p_2,p_3-2,p_4+2} - \frac{1}{2}(p_4 + 1)(p_4 + 2) a_{p_2-3,p_3,p_4+2} + \\
& + \frac{9}{4}(p_3 + 1)(p_3 + 2) a_{p_2,p_3+2,p_4-1} - \frac{9}{16}(p_3 + 1)(p_3 + 2) a_{p_2-2,p_3+2,p_4}. \quad (17)
\end{aligned}$$

The first line again contains terms where only the  $p_2$  index vary ( $p_3$  and  $p_4$  remain fixed in these expressions). The remaining terms are mixing terms, which are induced by the nonvanishing scalar products:  $\langle 2, 0, 0 | 0, 0, 1 \rangle \neq 0$ ,  $\langle 1, 2, 0 | 0, 0, 2 \rangle \neq 0$ ,  $\langle 1, 0, 1 | 0, 2, 0 \rangle \neq 0$  and  $\langle 3, 0, 0 | 0, 2, 0 \rangle \neq 0$ .

Summarizing, a general feature of the recursion relation eq.(15) is that only the first three terms describe a change in the  $p_2$  index of the  $a_{p_2,p_3,\dots,p_N}$  coefficients. Moreover, these terms have the structure of the Laguerre recursion relation <sup>2</sup> with  $2E$  being the argument of the polynomials and  $\sum_{k=3}^N k p_k + \frac{1}{2}(N^2 - 1) - 1$  playing the role of their index. The additional terms are responsible for the mixing between states with an equal number of quanta but constructed with different elementary bricks. However, keeping these indices as external parameters, one can solve eq.(17).

In sections 6 and 7 we will use these observations and the general theorems developed in Ref.[12] for the  $SU(3)$  model, to discuss the eigenvalues and eigenstates of  $H$  which solves the above recursion relations.

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<sup>2</sup>We define the Laguerre polynomials  $\mathcal{L}_m^\alpha(x)$  as the solutions of the differential equation  $xy'' + (\alpha + 1 - x)y' + ny = 0$  and the orthogonality relation  $\int_0^\infty \mathcal{L}_m^\alpha(x) \mathcal{L}_n^\alpha(x) x^\alpha e^{-x} dx = \delta_{mn}$ . The polynomials  $L_m^\alpha(x)$  are related to  $\mathcal{L}_m^\alpha(x)$  via  $L_m^\alpha(x) = \frac{\mathcal{L}_m^\alpha(x)}{\Gamma(m + \alpha + 1)}$ , where  $\Gamma(m)$  is the Euler gamma function,  $\Gamma(m + 1) = m!$  for  $m$  integer.

## 4.2 Fermionic sectors

Before we discuss the solutions of eq.(15) let us generalize the above treatment for the case of fermionic sectors. Since we have

$$[H, C^\dagger(n_B^\alpha, n_F, \alpha)] = \frac{1}{2} n_B^\alpha C^\dagger(n_B^\alpha, n_F, \alpha) - \frac{1}{2} [(aa), C^\dagger(n_B^\alpha, n_F, \alpha)] \quad (18)$$

we can rewrite the eigenequation of  $H$  as

$$\begin{aligned} & \sum_{\alpha=1}^{d^N_F} \sum_{\sum_{k=2}^N k p_k \leq N_{cut} - n_B^\alpha} a_{p_2, p_3, \dots, p_N}^\alpha C^\dagger(n_B^\alpha, n_F, \alpha) \left( H + \frac{n_B^\alpha}{2} \right) |p_2, p_3, \dots, p_N\rangle \\ & - \frac{1}{2} \sum_{\alpha=1}^{d^N_F} \sum_{\sum_{k=2}^N k p_k \leq N_{cut} - n_B^\alpha} a_{p_2, p_3, \dots, p_N}^\alpha [(aa), C^\dagger(n_B^\alpha, n_F, \alpha)] |p_2, p_3, \dots, p_N\rangle = 0. \end{aligned} \quad (19)$$

The action of the Hamiltonian on the basis states has been derived in the previous section. In order to obtain the recurrence relation we have to compute the commutators  $[(aa), C^\dagger(n_B^\alpha, n_F, \alpha)]$  for any fermionic brick and any  $N$ . We proceed in the same manner as was did for the  $SU(3)$  model[12]. For any fermionic brick  $C^\dagger(n_B^\alpha, n_F, \alpha)$ , the commutator with  $(aa)$  will be equal to a sum of  $n_B^\alpha$  terms, each of them being equal to the fermionic brick  $C^\dagger(n_B^\alpha, n_F, \alpha)$  with one of the bosonic creation operators substituted by an annihilation operator. We will write them as  $G_\alpha^t$ , where the index  $t$  goes from 1 to  $n_B^\alpha$ ,

$$[(aa), C^\dagger(n_B^\alpha, n_F, \alpha)] = \sum_{t=1}^{n_B^\alpha} G_\alpha^t. \quad (20)$$

The operators  $G_i^t$  should be now pushed on the right through the creation operators  $\prod_{k=2}^N (a^{\dagger k})^{p_k}$ . We get,

$$\forall_t [G_\alpha^t, \prod_{k=2}^N (a^{\dagger k})^{p_k}] = \sum_{j=2}^N \left( \prod_{k=2}^{j-1} (a^{\dagger k})^{p_k} \right) [G_\alpha^t, (a^{\dagger j})^{p_j}] \left( \prod_{k=j+1}^N (a^{\dagger k})^{p_k} \right) \quad (21)$$

$G_\alpha^t$  contains exactly one bosonic annihilation operators, therefore we can write

$$\forall_t [G_\alpha^t, (a^{\dagger j})^{p_j}] = p_j (a^{\dagger j})^{p_j-1} [G_\alpha^t, (a^{\dagger j})]. \quad (22)$$



For  $j = 2$ , we can replace  $[G_\alpha^t, (a^\dagger a^\dagger)]$  in the above expression by  $C^\dagger(n_B^\alpha, n_F, \alpha)$ . Thus, we get the general form of the recurrence relation,

$$\begin{aligned}
& \sum_{\alpha=1}^{d^{n_F}} \sum_{\sum_{k=2}^N k p_k \leq N_{cut} - n_B^\alpha} \left\{ \left( a_{p_2-1, p_3, \dots, p_N}^\alpha + \right. \right. \\
& \quad \left. \left. - \left( 2p_2 + \sum_{k=3}^N k p_k + \frac{1}{2}(N^2 - 1) + n_B^\alpha - 2E \right) a_{p_2, p_3, \dots, p_N}^\alpha \right. \right. \\
& \quad \left. \left. + (p_2 + 1) \left( p_2 + \sum_{k=3}^N k p_k + \frac{1}{2}(N^2 - 1) + n_B^\alpha \right) a_{p_2+1, p_3, \dots, p_N}^\alpha \right) |\alpha, n_F; p_2, p_3, \dots, p_N\rangle \right. \\
& \quad \left. + \sum_{j=3}^N \left( \frac{p_j(p_j - 1)}{4} j^2 \left( a_{p_2, \dots, p_j+2, \dots, p_{2j-2}-1, \dots, p_N}^\alpha - \frac{1}{N} a_{p_2, \dots, p_{j-1}-2, p_j+2, \dots, p_N}^\alpha \right) \right. \right. \\
& \quad \left. \left. + p_j j \frac{N}{4} \left( 1 - \frac{j-1}{N^2} \right) a_{p_2, \dots, p_{j-2}-1, p_{j-1}, p_j+1, \dots, p_N}^\alpha + \right. \right. \\
& \quad \left. \left. + \frac{p_j j}{4} \sum_{t=2}^{j-4} a_{p_2, \dots, p_t-1, \dots, p_{j-2-t}-1, \dots, p_j+1, \dots, p_N}^\alpha + \right. \right. \\
& \quad \left. \left. + p_j j \sum_{s=j+1}^N \frac{p_s s}{2} \left( a_{p_2, \dots, p_j+1, \dots, p_s+1, \dots, p_{j+s-2}-1, \dots, p_N}^\alpha + \right. \right. \right. \\
& \quad \left. \left. \left. - \frac{1}{N} a_{p_2, \dots, p_{j-1}-1, p_j+1, \dots, p_{s-1}-1, p_s+1, \dots, p_N}^\alpha \right) \right) |\alpha, n_F; p_2, \dots, p_N\rangle \right. \\
& \quad \left. + a_{p_2, p_3, \dots, p_N}^\alpha \sum_{t=1}^{n_B^\alpha} \left( \sum_{k=3}^N p_k [G_\alpha^t, (a^\dagger)^k] |p_2, p_3, \dots, p_j - 1, \dots, p_N\rangle + \right. \right. \\
& \quad \left. \left. + \left( \prod_{k=2}^N (a^\dagger)^{p_k} \right) G_\alpha^t |0\rangle \right) \right\} = 0. \quad (23)
\end{aligned}$$

This recursion relation can be divided into three parts. The first part (first three terms contained in a parenthesis) is diagonal in all indices except  $p_2$ , this includes also the fermionic index  $\alpha$ . Again, it has the structure of the Laguerre polynomials recursion relation with  $2E$  playing the role of the argument of these polynomials. The second part (four terms contained in a parenthesis) mixes the indices  $p_j$ ,  $2 \leq j \leq N$ . However, it is still diagonal in the fermionic index  $\alpha$ . The third part contains terms which mix different values of the  $\alpha$  index. It can be argued that this latter part does

not contain terms proportional to the  $C^\dagger(n_B^\alpha, n_F, \alpha)$  fermionic brick, hence it really corresponds to a mixing part.

## 5 Grouping into families

One of the most important conclusion of this note is the observation that the solutions, both bosonic as well as fermionic, group into disjoint sets. This is true for the models with arbitrary  $N$ . Such sets of solutions were called *families* in the study of the  $SU(3)$  model[12]. In this latter case, each solution belonged to an unique family. Families were labeled by a single integer denoting the maximal number of cubic bricks which appeared in the decomposition of the solutions from this family in the Fock basis. Basing on the form of the recursion relations eqs.(15) and (23) we now argue that these remarks can be generalized to the case of the  $SU(N)$  model with arbitrary  $N$ . This is by no means a strick proof; in order to obtain a specific solution one should apply a general theorem presented in Ref.[12]. Below, we just sketch the argument.

We start with the discussion of the purely bosonic recursion relation. Let us note that at *finite* cut-off there always exists one set of coefficients  $a_{p_2, p_3, \dots, p_N}$  which recursion relation is not coupled to any other. This is because by cutting the Fock basis we do not consider states with sufficiently large number of quanta. In order to be more specific let us fix the cut-off to be equal to  $N_{cut}$ . We choose a set of integers  $p_3^{max}, p_4^{max}, \dots, p_N^{max}$ , such that  $\sum_{k=3}^N k p_k^{max} = N_{cut}$ . One of the possible choices is  $p_4^{max} = p_5^{max} = \dots = p_N^{max} = 0$  and  $3p_3^{max} = N_{cut}$ . Other choices are independent and lead to the same conclusions. Consider now the recursion relation for  $a_{p_2, p_3^{max}, p_4^{max}, \dots, p_N^{max}}$ . There is only one equation which reads

$$-(N_{cut} + \frac{1}{2}(N^2 - 1) - 2E)a_{0, \frac{1}{3}N_{cut}, 0, \dots, 0} = 0. \quad (24)$$

Other coefficients  $a_{p_2, \frac{1}{3}N_{cut}, 0, \dots, 0}$  are absent because they contain too many quanta. From eq.(24) follow two possibilities. Either  $2E = N_{cut} + \frac{1}{2}(N^2 - 1)$  and then  $a_{0, \frac{1}{3}N_{cut}, 0, \dots, 0}$  can be arbitrary or  $2E \neq N_{cut} + \frac{1}{2}(N^2 - 1)$  in which case  $a_{0, \frac{1}{3}N_{cut}, 0, \dots, 0}$  must vanish. The condition for  $E$  can be also rewritten in the form,

$$L_1^{N_{cut} + \frac{1}{2}(N^2 - 1) - 1}(2E) = 0 \quad \text{or} \quad L_1^{N_{cut} + \frac{1}{2}(N^2 - 1) - 1}(2E) \neq 0. \quad (25)$$

Now, if the coefficient  $a_{0, \frac{1}{3}N_{cut}, 0, \dots, 0}$  vanishes (we assume that so do the coefficients corresponding to other choices of  $p_3^{max}, p_4^{max}, \dots, p_N^{max}$ , such that

$\sum_{k=3}^N kp_k^{max} = N_{cut}$ ), then the recursion relation for the coefficients  $a_{p_2, \frac{1}{3}N_{cut}-2, 0, \dots, 0}$  has no mixing terms and can be easily solved. The solution yields a new quantization condition of the form similar to eq.(25), namely,

$$L_4^{N_{cut} + \frac{1}{2}(N^2-1)-7}(2E) = 0. \quad (26)$$

For  $E$  satisfying the condition eq.(26), nontrivial values for the coefficients  $a_{p_2, \frac{1}{3}N_{cut}-2, 0, \dots, 0}$  are possible. For  $E$  which does not satisfy the condition eq.(26), the corresponding coefficients must vanish, hence yielding the recursion relation for another set of coefficients without mixing terms. In this way the families of solutions arise. For every  $E$  satisfying any of the quantization conditions a new solution appear. All solutions coming from a single quantization condition have similar properties; in particular, the amplitudes in their decomposition in the Fock basis are given by Laguerre polynomials with the same index. Every solution can be unambiguously denoted by its energy  $E$  and a set of integer numbers  $p_3^{max}, p_4^{max}, \dots, p_N^{max}$ , where  $p_t^{max}$  denotes the maximal power of the elementary brick ( $a^{\dagger t}$ ) in the decomposition of the eigenstate in the basis. It is a natural generalization of the results obtained for the  $SU(3)$  model, in which case the solutions were labeled by  $E$  and a single integer  $p_3^{max}$ . Hence, for example for the model with  $SU(4)$  gauge symmetry, the families are labeled by two integers  $p_3^{max}$  and  $p_4^{max}$ .

Summarizing, for a given cut-off  $N_{cut}$  the eigenenergies are given by a set of quantization conditions, all of which have the following form,

$$L^{\sum_{k=3}^N kp_k + \frac{1}{2}(N^2-1)-1}_{\lfloor \frac{1}{2}(N_{cut} - (\sum_{k=3}^N kp_k) \rfloor + 1)}(2E) = 0. \quad (27)$$

Each set of numbers  $\{p_k\}$  corresponds to a quantization condition and yields a new family of solutions. The index of Laguerre polynomials in eq.(27) have the structure

$$\gamma = \sum_{k=3}^N kp_k + \frac{1}{2}(N^2 - 1) - 1. \quad (28)$$

All eigensolutions belonging to such family have decomposition coefficients in the Fock basis given by the Laguerre polynomials of index given by eq.(28) and argument  $2E$ .

By comparing the structure of the fermionic recursion relation eq.(23) to the bosonic recursion relation eq.(15) it is straightforward to generalize the above remarks to the fermionic sectors. Indeed, eq.(23) have a part corresponding to the recursion relation in the  $p_2$  index and a part corresponding to the mixing. One can verify that in the fermionic case the solutions of

similar properties also group into families. They can be characterized by the index of Laguerre polynomials of the structure

$$\gamma = \sum_{k=3}^N k p_k + \frac{1}{2}(N^2 - 1) - 1 + n_B^\alpha, \quad (29)$$

where  $n_B^\alpha$  describes the properties of the fermionic brick used in the decomposition of solutions in the Fock basis. The families are labeled by the set of integers  $\{p_k\}$  and the index  $\alpha$ .

Grouping of solutions into families has two important consequences. First, as it is obvious from the above discussion, it allows to write explicitly the expressions for the spectra in all sectors for all  $SU(N)$  models. This follows from the fact that each family has its own quantization condition, all of them having a similar structure. We discuss this in more details in section 6. Second, one is able to consider solutions belonging to a single family, and therefore in the simplest cases write the solutions explicitly for any  $N$ . This feature is discussed in section 7.

## 6 Spectra

In this section we present closed formulae describing the spectra of the SYMQM models with arbitrary  $SU(N)$  gauge symmetry.

### 6.1 Bosonic sectors

For a given cut-off  $N_{cut}$ , we can define a polynomial  $\Theta_{N_{cut}}^{n_F=0}(N, E)$ , whose zeros correspond to all eigenenergies of the cut Hamiltonian operator in the bosonic sector,

$$\left\{E\right\}_{N_{cut}}^{n_F=0} = \left\{\Theta_{N_{cut}}^{n_F=0}(N, E) = 0\right\}. \quad (30)$$

$\Theta_{N_{cut}}^{n_F=0}(N, E)$  can be expressed as a product of quantization conditions of all nonempty families of solutions. For example, for the  $SU(4)$  model we have

$$\Theta_{N_{cut}}^{n_F=0}(4, E) = \prod_{t=0}^{\lfloor \frac{1}{3}N_{cut} \rfloor} \left( \prod_{k=0}^{\lfloor \frac{1}{4}(N_{cut}-3t) \rfloor} L_{\lfloor \frac{1}{2}(N_{cut}-3t-4k) \rfloor + 1}^{3t+4k+\frac{1}{2}(16-1)-1}(2E) \right), \quad (31)$$

whereas for the  $SU(5)$  model

$$\begin{aligned} \Theta_{N_{cut}}^{n_F=0}(5, E) &= \\ &= \prod_{t=0}^{\lfloor \frac{1}{3}N_{cut} \rfloor} \left( \prod_{k=0}^{\lfloor \frac{1}{4}(N_{cut}-3t) \rfloor} \left( \prod_{s=0}^{\lfloor \frac{1}{5}(N_{cut}-3t-4k) \rfloor} L_{\lfloor \frac{1}{2}(N_{cut}-3t-4k-5s) \rfloor + 1}^{3t+4k+5s+\frac{1}{2}(25-1)-1} (2E) \right) \right). \end{aligned} \quad (32)$$

These formulas were checked with independent numerical calculations which exploited a recursive algorithm[6]. With both methods results for  $N_{cut} \leq 20$  were obtained and agreed exactly.

From the recursion relation eq.(15) follows a general formula for the polynomial  $\Theta_{N_{cut}}^{n_F=0}(N, E)$  for any  $N$ ,

$$\Theta_{N_{cut}}^{n_F=0}(N, E) = \prod_{i=3}^N \left( \prod_{p_i=0}^{\lfloor \frac{1}{i}(N_{cut} - (\sum_{k=3}^{i-1} kp_k)) \rfloor} L_{\lfloor \frac{1}{2}(N_{cut} - (\sum_{k=3}^N kp_k) + \frac{1}{2}(N^2-1)-1) \rfloor + 1}^{\sum_{k=3}^N kp_k + \frac{1}{2}(N^2-1)-1} (2E) \right). \quad (33)$$

Once the whole spectrum can be calculated thanks to eq.(33) one should be able to compute the Witten index or the microcanonical partition function of the SYMQM systems.

From the properties of the Laguerre polynomials, one can conclude that the smallest eigenvalues will belong to the family with the smallest index and the biggest order. These conditions can be satisfied by setting  $\sum_k kp_k = 0$ . Hence, the smallest eigenenergies will belong to the simplest family  $\{p_k\} = \{0, 0, \dots, 0\}$ , the one without any mixing.

## 6.2 Fermionic sectors

The generalization of the above results to the fermionic sectors is immediate. The polynomial  $\Theta_{N_{cut}}^{n_F}(N, E)$  valid in all sector for any  $SU(N)$  model can be obtained, basing on the recursion relation eq.(23), in the form

$$\begin{aligned} \Theta_{N_{cut}}^{n_F}(N, E) &= \prod_{\alpha=1}^{d^{n_F}(N)} \left\{ \prod_{i=3}^N \left( \prod_{p_i=0}^{\lfloor \frac{1}{i}(N_{cut} - (\sum_{k=3}^{i-1} kp_k) - n_B^\alpha(N)) \rfloor} L_{\lfloor \frac{1}{2}(N_{cut} - (\sum_{k=3}^N kp_k) - n_B^\alpha(N)) \rfloor + 1}^{(\sum_{k=3}^N kp_k) + \frac{1}{2}(N^2-1)-1+n_B^\alpha(N)} (2E) \right) \right\}, \end{aligned} \quad (34)$$

where the numbers  $d^{n_F}(N)$  and  $n_B^\alpha(N)$  depend on  $N$ .

### 6.3 Continuum limit

The continuum limit can be simply obtained by taking the limit  $N_{cut} \rightarrow \infty$  in the expressions eq.(33) and eq.(34). The eigenvalues form a dense subset of the positive real numbers. Each eigenenergy is infinitely degenerate. This does not concern the non-degenerate supersymmetric vacua which will be shown when discussing the wave-functions of the supersymmetric vacua in the next section. The continuum limit involving the scaling law in the spirit of Refs.[10, 11] will be discussed elsewhere.

## 7 Discussion of the simplest solutions

In Ref.[13] Trzetrzelewski has formulated an algorithm for finding the sectors of the SYMQM systems where the supersymmetric vacua are located. Using the recursion relation eq.(15) we can construct the wavefunctions of these vacua by taking the limit of  $E \rightarrow 0$  of our solutions. However, in order to be able to take this limit, first one has to consider the solutions with  $N_{cut} \rightarrow \infty$ . We discuss these two limits below.

### 7.1 Solutions at finite cut-off

Let us start by describing the solutions at finite cut-off. The most general solution has the following form

$$\begin{aligned}
|E, p_3, p_4, \dots, p_N\rangle^{n_F=0} = & \\
= e^{-E} \sum_{n=0}^{d-1} L_n^{\frac{1}{2}(N^2-1)+\sum_{s=3}^N k p_k-1} (2E) & \left( |n, p_3, p_4, \dots, p_N\rangle + \right. \\
+ \sum_{t_3, t_4, \dots, t_N=1}^{p_3, p_4, \dots, p_N} A_{t_3, t_4, \dots, t_N}^{p_3, p_4, \dots, p_N} & |n + \sum_{s=3}^N s t_s, p_3 - t_3, p_4 - t_4, \dots, p_N - t_N\rangle \Big). \quad (35)
\end{aligned}$$

The parameter  $d$  in the above sum is equal to the number of solutions belonging to the family denoted by  $\{p_3, p_4, \dots, p_N\}$  at finite cut-off. Obviously  $d$  must depend on the cut-off,  $d = d(N_{cut})$ . The coefficients  $A_{t_3, t_4, \dots, t_N}^{p_3, p_4, \dots, p_N}$  must be determined from the recursion relation eq.(15). These amplitudes depend only on the set of integers  $\{p_k\}$ ; especially, they do not depend on  $N_{cut}$ .

The simplest solutions belong to the family  $\{0, 0, \dots, 0\}$ . In this case  $A_{t_3, t_4, \dots, t_N}^{p_3, p_4, \dots, p_N} \equiv 0$ , i.e. the solutions have no mixing and hence are only built

out of bilinear bosonic bricks. They can be written as

$$|E_m, 0, \dots, 0\rangle^{n_F=0} = e^{-E_m} \sum_{n=0}^{d_0-1} L_n^{\frac{1}{2}(N^2-1)-1} (2E_m) |n, 0, \dots, 0\rangle, \quad 1 \leq m \leq d_0, \quad (36)$$

where  $d_0 = \lfloor \frac{N_{cut}}{2} \rfloor + 1$ , and  $E_m$  are such that  $L_{d_0}^{\frac{1}{2}(N^2-1)-1} (2E_m) = 0$ . As an example of solutions belonging to more complex families, we present solutions from the family  $\{2, 0, \dots, 0\}$  valid for any  $N > 3$ . They read

$$|E_m, 2, 0, \dots, 0\rangle^{n_F=0} = e^{-E_m} \sum_{n=0}^{d_1-1} L_n^{\frac{1}{2}(N^2-1)+5} (2E_m) \left( |n, 2, 0, \dots, 0\rangle + \right. \\ \left. - \frac{18}{N} \frac{1}{24 + 6(N^2 - 1)} |n + 3, 0, 0, \dots, 0\rangle \right), \quad 1 \leq m \leq d_0. \quad (37)$$

$d_1$  denotes the number of solutions of this type for a given cut-off and is given by  $d_1 = \lfloor \frac{1}{2}(N_{cut} - 6) \rfloor$  and  $E_m$  are such that  $L_{d_1}^{\frac{1}{2}(N^2-1)+5} (2E_m) = 0$  this time.

States from other sectors can be easily obtained by acting with the fermionic bricks on eqs.(36) or (37). If one uses purely fermionic bricks, then no mixing between fermionic bricks appears. Therefore, the states presented below are solutions of the recursion relation eq.(23),

$$|E_m, 0, \dots, 0\rangle^{n_F} = \\ = e^{-E_m} \sum_{n=0}^{d_0-1} L_n^{\frac{1}{2}(N^2-1)-1} (2E_m) (f^{\dagger 3})^{i_3} (f^{\dagger 5})^{i_5} \dots (f^{\dagger(2N-1)})^{i_{2N-1}} |n, 0, \dots, 0\rangle, \quad (38)$$

where  $i_3, i_5, \dots, i_{2N-1} \in \{0, 1\}$  and  $n_F = \sum_{k=2}^N (2k-1) i_{2k-1}$ . Such states can be found in the spectrum of all  $SU(N)$  models (see also Ref.[13]). Similarly,

$$|E_m, 2, 0, \dots, 0\rangle^{n_F} = \\ = e^{-E_m} \sum_{n=0}^{d_1-1} L_n^{\frac{1}{2}(N^2-1)+5} (2E_m) (f^{\dagger 3})^{i_3} (f^{\dagger 5})^{i_5} \dots (f^{\dagger(2N-1)})^{i_{2N-1}} \times \\ \times \left( |n, 2, 0, \dots, 0\rangle - \frac{18}{N} \frac{1}{24 + 6(N^2 - 1)} |n + 3, 0, 0, \dots, 0\rangle \right), \quad (39)$$

with  $n_F = \sum_{k=2}^N (2k-1) i_{2k-1}$ .

In this way expressions for increasingly complicated solutions can be obtained.

## 7.2 Continuum limit

Similarly to the case of the  $SU(3)$  model, the continuum limit can be easily obtained from the finite-cut-off solutions. The only quantity dependent on  $N_{cut}$  in eqs.(36) and (38) and in eqs.(37) and (39) are the upper limits of the sums,  $d_0$  and  $d_1$ , and the eigenenergies  $E_m$ . Hence, the continuum limit of these solutions can be simply obtained by extending the sums to infinity,

$$\begin{aligned} |E, 0, \dots, 0\rangle^{n_F=0} &= e^{-E} \sum_{n=0}^{\infty} L_n^{\frac{1}{2}(N^2-1)-1}(2E) |n, 0, \dots, 0\rangle, \\ |E, 2, 0, \dots, 0\rangle^{n_F=0} &= e^{-E} \sum_{n=0}^{\infty} L_n^{\frac{1}{2}(N^2-1)+5}(2E) \left( |n, 2, 0, \dots, 0\rangle + \right. \\ &\quad \left. - \frac{18}{N} \frac{1}{24 + 6(N^2 - 1)} |n + 3, 0, 0, \dots, 0\rangle \right), \end{aligned}$$

and adequately for the fermionic solutions. In the continuum limit the set of eigenenergies,  $\{E\}$ , is dense in the set of real, positive numbers.

## 7.3 Normalization

The normalization of the presented states can be calculated for the simplest solutions eqs.(36) and (38). For the continuum states we have

$$\begin{aligned} \langle E|E'\rangle &= \frac{\Delta e^{-E-E'}}{\Gamma(\frac{1}{2}(N^2-1))} \times \\ &\times \lim_{z \rightarrow 1^-} \frac{1}{1-z} e^{\frac{-2z}{1-z}(E+E')} (4EE'z)^{-\frac{1}{2}(\frac{1}{2}(N^2-1)-1)} I_{\frac{1}{2}(N^2-1)-1} \left( \frac{4\sqrt{EE'}z}{1-z} \right), \end{aligned} \quad (40)$$

where

$$\Delta = \begin{cases} \langle 0|0\rangle = 1 \text{ in the bosonic sector} \\ \langle 0|(f^{2N-1})_{i_{2N-1}} \dots (f^5)_{i_5} (f^3)_{i_3} (f^{\dagger 3})_{i_3} (f^{\dagger 5})_{i_5} \dots (f^{\dagger(2N-1)})_{i_{2N-1}} |0\rangle \\ \text{in the sector with } n_F = \sum_{k=2}^N (2k-1)_{i_{2k-1}}. \end{cases}$$

Expressing  $z$  as  $z = 1 - 4\epsilon$  we get

$$\langle E|E'\rangle = \frac{\Delta}{2\sqrt{\pi}} \frac{e^{E+E'-2\sqrt{EE'}}}{\Gamma(\frac{1}{2}(N^2-1))} (4EE')^{-\frac{1}{2}(\frac{1}{2}(N^2-1)-1)-\frac{1}{4}} \lim_{\epsilon \rightarrow 0^+} \frac{1}{\sqrt{4\epsilon}} e^{-\frac{(\sqrt{2E}-\sqrt{2E'})^2}{4\epsilon}}. \quad (41)$$



Exploiting the well-known representation of Dirac delta distribution we have

$$\begin{aligned}\langle E|E'\rangle &= \frac{\Delta}{2\sqrt{\pi}} \frac{e^{E+E'-2\sqrt{EE'}}}{\Gamma(\frac{1}{2}(N^2-1))} (4EE')^{-\frac{1}{4}(N^2-1)+\frac{1}{4}} \frac{\sqrt{\pi}}{2} \delta(\sqrt{2E}-\sqrt{2E'}) \\ &= \frac{\Delta}{4\Gamma(\frac{1}{2}(N^2-1))} (2E)^{-\frac{1}{2}(N^2-2)} \delta(\sqrt{2E}-\sqrt{2E'}).\end{aligned}\quad (42)$$

Hence, the continuum solutions belonging to these simplest solutions are orthogonal to each other and normalized as plane-waves. It may be explicitly checked that they are also orthogonal to solutions of any other family.

The presence of the factor  $(2E)^{-\frac{1}{2}(N^2-2)}$  may be linked with the jacobian of the change of variables from the 'cartesian' variables to the 'spherical' variables. One can think of a set of 'Fourier transformed' degrees of freedom denoted by  $k_A$  corresponding to the original set of degrees of freedom  $\phi_A$ . The normalization factor then depends only on the 'radial' variable,  $(2E)^2 = \sum_A k_A^2$ . The explicit form of the 'Fourier' transformation appropriate for the  $SU(N)$  manifolds is not known. It is therefore surprising that, the  $SU(N)$  manifold being  $N^2 - 1$  dimensional, the mentioned factor in eq.(42) corresponds exactly to the jacobian of the change of variables in a  $N^2 - 1$  dimensional Euclidean space.

## 7.4 Vacuum solutions

Imposing the normalization of all solutions according to eq.(42) has an important consequence. It turns out that solutions belonging to more complicated families  $\{p_3, p_4, \dots, p_N; \alpha\}$  with at least one of  $p_i > 0$  acquire an normalization factor of the form  $(2E)^\gamma$ , where  $\gamma$  is a positive real number. This implies, that in the limit of  $E \rightarrow 0$ , these solutions vanish. Hence, the supersymmetric vacua can exist only in the sectors where the simplest families  $\{0, 0, \dots, 0; \alpha\}$  can be constructed. These are exactly the same sectors as those found in Ref.[13] where these conclusions were reached from the cohomology of the  $SU(N)$  groups point of view.

Finally, let us note that the fundamental theorem of supersymmetry, namely

$$Q|\text{vacuum}\rangle = Q^\dagger|\text{vacuum}\rangle = 0 \Leftrightarrow \langle \text{vacuum}|H|\text{vacuum}\rangle = 0, \quad (43)$$

may not hold when the vacuum state,  $|\text{vacuum}\rangle$ , is not normalizable. Indeed, in the cases discussed above,  $Q|\text{vacuum}\rangle \neq 0$  since the state  $Q|\text{vacuum}\rangle$  exists and can be calculated. However, with the normalization eq.(42) the state  $Q|\text{vacuum}\rangle$  has zero norm. In this situation, eq.(43) is valid and all the vacua are the true, non-degenerate, supersymmetric ground states.

## 7.5 Completeness

We emphasized the fact that solutions can be divided into families. The transformation from the Fock states to the solutions of a given family was shown to be non-degenerate[12]. Hence, one can show that the transformation of the entire Fock basis onto the set of all solutions is also non-degenerate. Therefore, since the Fock basis was shown to span the entire Hilbert space of the SYMQM models, equivalently the set of all solutions is also complete.

## 8 Conclusions

In this work we generalized the solutions, which were recently derived for the  $SU(3)$  model, to the general case of models with  $SU(N)$  gauge groups. The new solutions have all the required properties: they are orthogonal, complete and normalized according to the plane-wave normalization. Hence, we obtained the correct  $SU(N)$  generalization of the Claudson-Halpern solutions derived for the  $SU(2)$  model.

After deriving the recursion relations for the amplitudes in the decomposition of the eigenstates in the Fock basis we discussed the properties of their solutions. We have emphasized that these solutions group into disjoint sets called families. Within each family the possible eigenenergies can be calculated from a single quantization condition which involves a Laguerre polynomial with a particular index  $\alpha$ . Moreover, we argued that the amplitudes of all solutions belonging to a given family are given by Laguerre polynomials with the same index  $\alpha$ .

In section 6 we provided closed formulae for the spectra of the studied models. These expressions are valid for any cut-off  $N_{cut}$  and for any gauge group  $SU(N)$ . They may be subsequently used in the studies of the thermodynamics of the SYMQM models (for a recent article on the thermodynamics of higher dimensional SYMQM models see Ref[14]).

In section 7 we discussed the properties of the simplest solutions. We explicitly proved the orthogonality of solutions belonging to the  $\{0, \dots, 0\}$  family as well as determined the normalization factors of these solutions. Consequently we could confirm the construction of supersymmetric vacua presented in Ref.[13] and we provided their correct wave-functions. However, due to the described normalization of our solutions, we did not have to impose any compactification as was done in Ref.[13].

Our results can be immediately used in the studies of the large- $N$  limit of the SYMQM models. The large- $N$  limit of systems in a Fock representation

was studied by Thorn[15] and was recently reexamined in the context of a supersymmetric quantum mechanical model in Refs.[16, 17, 18]. Additionally, the large- $N$  limit of wave-functions of higher dimensional SYMQM are particularly interesting since they can have an interpretation in the context of supermembrane theory [7]. The expressions derived in section 7 with their explicit  $N$  dependence may provide some new insight on this limit.

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## A Calculations leading to eq.(14)

In order to evaluate the action of the  $(aa)$  operator one has to move it through all operators composing the basis state to the point when it hits the Fock vacuum. Hence

$$(aa)|p_2, p_3, p_4, \dots, p_N\rangle = \sum_{j=2}^N \left( \prod_{i=2}^{j-1} (a^{\dagger i})^{p_i} \right) \left[ (aa), (a^{\dagger j})^{p_j} \right] \left( \prod_{i=j+1}^N (a^{\dagger i})^{p_i} \right) |0\rangle \quad (44)$$

We need the following commutators

$$\left[ (aa), (a^{\dagger n}) \right] = n(a^{\dagger n-1}a) + \frac{nN}{2} \left( \frac{1}{2} - \frac{n-1}{2N^2} \right) (a^{\dagger n-2}) + \frac{n}{4} \sum_{j=2}^{n-4} (a^{\dagger j}) (a^{\dagger n-2-j}). \quad (45)$$

Hence,

$$\begin{aligned} \left[ (aa), (a^{\dagger n})^m \right] &= \frac{1}{4} m(m-1) n^2 (a^{\dagger n})^{m-2} (a^{\dagger 2n-2}) + \\ &\quad - \frac{1}{4} m(m-1) \frac{n^2}{N} (a^{\dagger n})^{m-2} (a^{\dagger n-1})^2 + mn (a^{\dagger n})^{m-1} (a^{\dagger n-1}a) + \\ &\quad + mn \frac{N}{2} \left( \frac{1}{2} - \frac{n-1}{2N^2} \right) (a^{\dagger n-2}) (a^{\dagger n})^{m-1} + \frac{mn}{4} \sum_{j=2}^{n-4} (a^{\dagger j}) (a^{\dagger n-2-j}) (a^{\dagger n})^{m-1} \end{aligned} \quad (46)$$

and for any positive integers  $A$  and  $B$

$$\left[ (a^{\dagger n}a), (a^{\dagger m})^k \right] = k \left( \frac{m}{2} (a^{\dagger n+m-1}) - \frac{m}{2N} (a^{\dagger n}) (a^{\dagger m-1}) \right) (a^{\dagger m})^{k-1} \quad (47)$$

$$\begin{aligned}
& \left[ (a^{\dagger n-1} a), \left( \prod_{i=n+1}^N (a^{\dagger i})^{p_i} \right) \right] = \\
& = \left( \prod_{i=n+1}^N (a^{\dagger i})^{p_i} \right) \sum_{t=n}^{N-1} \frac{p_{i+1}(i+1)}{2} \left( \frac{(a^{\dagger n+i-1})}{(a^{\dagger i+1})} - \frac{1}{N} \frac{(a^{\dagger n-1})(a^{\dagger i})}{(a^{\dagger i+1})} \right) \quad (48)
\end{aligned}$$

where we have introduced the notation

$$\prod_{i=A}^B \frac{(a^{\dagger i})^{p_i}}{(a^{\dagger t})^n} \equiv \left( \prod_{i=A}^{t-1} (a^{\dagger i})^{p_i} \right) (a^{\dagger t})^{p_t-n} \left( \prod_{i=t+1}^B (a^{\dagger i})^{p_i} \right). \quad (49)$$

Therefore,

$$\begin{aligned}
& \left[ (aa), \left( \prod_{j=2}^N (a^{\dagger j})^{p_j} \right) \right] |0\rangle = \\
& = \left[ p_2(p_2 + \frac{1}{2}(N^2 - 1) - 1 \sum_{s=3}^N s p_s) \right] \frac{\left( \prod_{j=2}^N (a^{\dagger j})^{p_j} \right)}{(a^{\dagger 2})} |0\rangle + \\
& + \sum_{j=3}^N \left[ \frac{j^2 p_j (p_j - 1)}{4} ((a^{\dagger 2j-2}) - \frac{1}{N} (a^{\dagger j-1})^2) + \frac{j p_j}{4} \sum_{t=2}^{j-4} (a^{\dagger t}) (a^{\dagger j}) (a^{\dagger j-2-t}) + \right. \\
& + \frac{N j p_j}{4} (1 - \frac{j-1}{N^2}) (a^{\dagger j}) (a^{\dagger j-2}) + \sum_{s=j+1}^N \frac{j p_j s p_s}{2} \left( \frac{(a^{\dagger j+s-2})(a^{\dagger j})}{(a^{\dagger s})} + \right. \\
& \left. \left. - \frac{1}{N} \frac{(a^{\dagger j-1})(a^{\dagger j})(a^{\dagger s-1})}{(a^{\dagger s})} \right) \right] \frac{\left( \prod_{j=2}^N (a^{\dagger j})^{p_j} \right)}{(a^{\dagger j})^2} |0\rangle. \quad (50)
\end{aligned}$$

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